The Quantum Hydrodynamics of the Sutherland Model

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Abstract

We show that the form of the chiral condition found by Abanov *et al.* in the quantum hydrodyamics of the Sutherland model arises because there are two distinct inner products with respect to which the chiral Hamitonian hermitian, but only one with respect to which the full, non-chiral, Hamiltonian is hermitian.

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I. INTRODUCTION

It has long been understood that there is a close relationship between the one-dimensional Luttinger-Thirring model [1] and the low energy edge modes of two-dimensional Laughlin-state fractional quantum Hall [2] fluids. The two systems have very similar ground-state wavefunctions and the edge-particle correlation functions can be computed from the Luttinger wavefunction [3]. The connection exists because the non-commuting x and y coordinates of the lowest Landau level can be regarded as the position and momentum co-ordinates of a fluid of one-dimensional fermions [4], and the boundary of the Hall fluid as its Fermi surface.

If we go beyond a linear approximation to the edge-state energies, the dimensional reduction becomes both more interesting and more complicated. The Hall fluid may be described by a Chern-Simons matrix model [5]. By adding boundary terms to the matrix model, Polychronakos showed that when a circular droplet of quantum Hall two-dimensional electron fluid is held in place by a harmonic $x^2 + y^2$ potential, the x-axis projected system becomes [6] a quantum Calogero model [7, 8] with the y^2 part of the two-dimensional potential providing the non-relativistic kinetic energy and the x^2 part providing a one-dimensional harmonic confining potential. When the Hall fluid is confined by a y^2 potential to a finite strip with x-periodic boundary conditions, the one-dimensional system becomes [9] the periodic Sutherland model [10, 11]. The ground-state wavefunction and low lying excitations still coincide with those of the Luttinger model, but the higher exited states are more complicated. There is a one-to-one mapping of the eigenstates of the harmonically confined quantum Hall systems onto the eigenstates of the Calogero-Sutherland models [12], and this mapping descends to the soliton and small amplitude wave solutions of the continuum classical hydrodymamics of the Calogero-Sutherland fluid [13].

One curious feature of this mapping is that we are looking at the two-dimensional quantum Hall fluid sideways-on, and see both its near and far edges superimposed. Although the two boundaries have their own independent edge modes that move in opposite directions, it is not easy to make a clean left-right separation in the projected one-dimensional quantum hydrodynamics [13]. Recently, however, Abanov *et al.*. [14] showed that the complicated non-local hydrodynamic equations were much simplified when expressed in terms of a dynamical field $u(z, t)$ that lives on a Schottky double constructed by gluing two copies of a

non-compact complex plane together along their boundaries. This ingenious reformulation, which depends on an unusual form of Hilbert transform, enabled them to find a condition linking the density and velocity under which only unidirectional motion is excited [15]. The definitions of the field and unconventional Hilbert transform in [14] are not at all obvious, however, and the way in which the chiral condition works seems almost magical.

The present paper is devoted to an alternative formulation of the quantum hydrodynamics that avoids the Schottky-double contour integrals that are the key element in [14, 15]. The non-obvious form of the chiral condition arises because there are two distinct inner products with respect to which the Jack-polynomial eigenfunctions of the Sutherland model are mutually orthogonal [16, 17]. The first of these is the one most often met with in the literature of symmetric functions. The second is the one that arises from the quantum mechanics. The chiral version of the Sutherland model is hermitian with respect to both these inner products. The full, non-chiral, version is hermitian only with respect to the second inner product. The mysterious terms that appear in [14, 15] are precisely the corrections required to make the "natural" chiral fields into operators that are hermitian with respect to the second product.

In section II we provide a brief account of the Sutherland model. In section III we review the application of the collective field formalism [18] to this model, and introduce the two inner products. In section IV we show how the difference between the two products manifests itself in the quantum hydrodynamics, and in section V how this difference is the origin of the complications in the chiral decomposition.

II. THE SUTHERLAND MODEL

We begin with a short review of the Sutherland model [10]. This consists of N particles moving on the unit circle with Hamiltonian

$$H_{\text{Sutherland}} = -\frac{1}{2}\sum_{i=1}^{N}\frac{\partial^2}{\partial\theta_i^2} + \frac{1}{4}\sum_{i<j}\frac{\lambda(\lambda-1)}{\sin^2(\theta_i-\theta_j)/2}. \tag{1}$$

We will restrict ourselves to the parameter range $\lambda > 1$, where the inter-particle interaction is repulsive. The potential is sufficiently singular that tunneling does not occur, and the particles retain their original order around the circle. The exchange statistics of the particles are therefore unimportant, but we will usually think of them as being fermions, as this is

their natural description in the limit $\lambda \to 1_+$.

The ground state wavefunction and energy may be found by means of a "supersymmetric quantum mechanics" trick. We set

$$\Delta = \prod_{i<j} 2\sin(\theta_i - \theta_j)/2, \tag{2}$$

and make use of the addition formula

$$\cot(x-y)\cot(y-z) + \cot(y-z)\cot(z-x) + \cot(z-x)\cot(x-y) = 1 \tag{3}$$

to write

$$\begin{aligned}
H &\equiv H_{\text{Sutherland}} - \frac{\lambda^2}{24}N(N^2-1) \\
&= -\frac{1}{2}\sum_i \frac{\partial^2}{\partial\theta_i^2} + \frac{1}{4}\sum_{i<j}\frac{\lambda(\lambda-1)}{\sin^2(\theta_i-\theta_j)/2} - \frac{\lambda^2}{24}N(N^2-1) \\
&= \frac{1}{2}\sum_i \left(-\frac{\partial}{\partial\theta_i} - \frac{\lambda}{2}\sum_{j\neq i}\cot(\theta_i-\theta_j)/2\right)\left(\frac{\partial}{\partial\theta_i} - \frac{\lambda}{2}\sum_{j\neq i}\cot(\theta_i-\theta_j)/2\right) \\
&= \frac{1}{2}\sum_i \left(-\frac{1}{\Delta^\lambda}\frac{\partial}{\partial\theta_i}\Delta^\lambda\right)\left(\Delta^\lambda\frac{\partial}{\partial\theta_i}\frac{1}{\Delta^\lambda}\right) \\
&= \frac{1}{2}\sum_{i=1}^N Q_i^\dagger Q_i.
\end{aligned} \tag{4}$$

It is now clear that

$$\Psi_0 = \Delta^\lambda(\theta) = \left(\prod_{i<j} 2\sin(\theta_i-\theta_j)/2.\right)^\lambda \tag{5}$$

satisfies $Q_i\Psi_0 = 0$ for all i, and is the unique zero-energy eigenfunction of H. The wavefunction Ψ_0 is therefore the ground state of the Sutherland model, and the ground-state energy is

$$E_0 = \frac{\lambda^2}{24}N(N^2-1). \tag{6}$$

This energy reduces for $\lambda = 1$ to the energy of an N-particle Fermi sea obeying periodic boundary conditions when N is odd and antiperiodic boundary conditions when N is even.

Now we seek wavefunctions of form $\Psi = \Delta^\lambda \Phi(z_1, \ldots, z_N)$, *i.e.* functions such that $H'\Phi = E'\Phi$, where

$$H' = -\frac{1}{2}\sum_{i=1}^N \frac{1}{\Delta^{2\lambda}}\frac{\partial}{\partial\theta_i}\Delta^{2\lambda}\frac{\partial}{\partial\theta_i}$$

$$
\begin{aligned}
&= -\frac{1}{2}\sum_i \frac{\partial^2}{\partial\theta_i^2} - \frac{\lambda}{2}\sum_{i,j\neq i}\cot(\theta_i-\theta_j)/2\frac{\partial}{\partial\theta_i} \\
&= -\frac{1}{2}\sum_i \frac{\partial^2}{\partial\theta_i^2} - \frac{\lambda}{2}\sum_{i<j}\cot(\theta_i-\theta_j)/2\left(\frac{\partial}{\partial\theta_i}-\frac{\partial}{\partial\theta_j}\right) \\
&= \frac{1}{2}\sum_i D_i^2 + \frac{\lambda}{2}\sum_{i<j}\frac{z_i+z_j}{z_i-z_j}(D_i-D_j).
\end{aligned}
\tag{7}
$$

In the last line we have set $z_i = \exp\{i\theta_i\}$ and $D_i = z_i\partial/\partial z_i = -i\partial/\partial\theta_i$. The Hamiltonian H' will be hermitian with respect to the "λ-Sutherland" inner product

$$
\langle\Phi_1|\Phi_2\rangle_{\text{Sutherland}} = \frac{1}{N!}\int_0^{2\pi}\cdots\int_0^{2\pi}\prod_{i=1}^N\frac{d\theta_i}{2\pi}|\Delta|^{2\lambda}\Phi_1^*\Phi_2. \tag{8}
$$

We initially only consider $\Phi(z_1,\ldots,z_N)$ that are symmetric polynomials in the z_i. These describe excitations near the $k\sim k_f$ Fermi point. Later we will worry about the z_i^{-1}'s that can be used for the $k\sim -k_f$ Fermi point.

Sutherland [11] considered, in particular, the action of H' on the monomial symmetric functions

$$
m_{\{\alpha\}}(z) = \sum z_{i_1}^{\alpha_1}z_{i_2}^{\alpha_2}\ldots z_{i_N}^{\alpha_N}, \tag{9}
$$

where

$$
\{\alpha\}\equiv\{\alpha_1,\alpha_2,\ldots,\alpha_N\},\quad \alpha_1\geq\alpha_2\geq\cdots\geq\alpha_N, \tag{10}
$$

and the sum is over all permutations of the labels i that result in distinct monomials.

We can represent the integer sequence α_i by a Young (or Ferrars) diagram with α_1 boxes in the first row, α_2 in the second, and so on, and think of it as a partition of the integer $|\{\alpha\}|\equiv\alpha_1+\alpha_2+\ldots$. We usually order partitions in reverse lexographic order, in which $\{\alpha\}>\{\beta\}$ if the first non zero difference $\alpha_i-\beta_i$ is positive. This is a *total* ordering: given two partitions one is greater than the other, or they are equal. An alternative ordering is *dominance ordering* in which $\{\alpha\}\succeq\{\beta\}$ if

$$
\sum_{i=1}^n\alpha_i\geq\sum_{i=1}^n\beta_i,\quad\forall n>0. \tag{11}
$$

Dominance is only a *partial* order (as is set inclusion) in that not all partitions are comparable. If $\{\alpha\}\succ\{\beta\}$, however, then $\{\alpha\}>\{\beta\}$. Sutherland showed that when the $m_{\{\alpha\}}$ are taken as a basis, then H' is represented by a matrix that is upper triangular with respect to dominance order. The eigenvalues of $H'$are therefore the diagonal elements of this

upper-triangular matrix. His result [11] is that the eigenvalues of $H_{\text{Sutherland}}$ can be written as

$$E_{\{\alpha\}} = E_0 + E'_{\{\alpha\}} = \frac{1}{2}\sum_i \xi_i^2. \tag{12}$$

where the *pseudomomenta* ξ_i are

$$\xi_i = \alpha_i + \lambda k_i^0, \tag{13}$$

with k_i^0 being the momenta of the Fermi sea of free fermions. The ground state has all the $\alpha_i = 0$, and so we recover the formula (6) for the ground-state energy.

The polynomial eigenfunctions of H' are the *Jack symmetric functions* $J_{\{\alpha\}}(z)$. They can, in principal be found by using $\langle \dots | \dots \rangle_{\text{Sutherland}}$ to apply the Gramm-Schmidt procedure to the reverse-lexographically-ordered monomial symmetric functions. It is a non-trivial result [16] that the only subtractions appearing in the orthogonalization process involve dominance-ordered $m_{\{\beta\}}$. Thus we obtain

$$J_{\{\alpha\}}(z) = m_{\{\alpha\}}(z) + \sum_{\{\beta\}\prec\{\alpha\}} K_{\{\alpha\}\{\beta\}} m_{\{\beta\}}(z), \tag{14}$$

and this condition, together with $\langle J_{\{\alpha\}} | J_{\{\beta\}} \rangle_{\text{Sutherland}} = 0$ when $\{\alpha\} \neq \{\beta\}$, serves to define the Jack functions uniquely.

When $\lambda = 1$, the Jack polynomials reduce to the Schur symmetric functions and the coefficients $K_{\{\alpha\}\{\beta\}}$ become Kostka numbers. Both the Schur and Jack functions are zero whenever the length $l(\{\alpha\})$ of the partition (the number of non-zero rows in the Young diagram) exceeds N.

III. COLLECTIVE FIELDS

We wish to describe the low energy and low momentum excitations of the Sutherland chain in terms of fluctuations in the particle density. We therefore change variables from $z_1, \dots z_N$ to $p_1, \dots, p_N$, where $p_n = \sum_i z_i^n$ are the Newton power-sum symmetric functions, and simultaneously the positive-momentum Fourier components of the density $\rho(\theta) = \sum_i^N \delta(\theta - \theta_i)$:

$$p_n = \int_0^{2\pi} e^{in\theta} \rho(\theta)\, d\theta. \tag{15}$$

We find that

$$D_i = z_i \frac{\partial}{\partial z_i} = z_i \sum_{n=1}^{N} \frac{\partial p_n}{\partial z_i} \frac{\partial}{\partial p_n} = \sum_{n=1}^{N} n z_i^n \frac{\partial}{\partial p_n}, \tag{16}$$

and from this obtain

$$\sum_i D_i^2 = \sum_{n=1}^{N} n^2 p_n \frac{\partial}{\partial p_n} + \sum_{m,n=1}^{N} nm\, p_{n+m} \frac{\partial}{\partial p_n} \frac{\partial}{\partial p_m}. \tag{17}$$

We now use

$$(z_i + z_j) \frac{z_i^n - z_j^n}{z_i - z_j} = z_i^n + 2 z_i^{n-1} z_j + \cdots + 2 z_i z_j^{n-1} + z_j^n \tag{18}$$

and, after some careful tracking of duplicated and omitted terms, obtain

$$\frac{\lambda}{2} \sum_{i \neq j} \frac{z_i + z_j}{z_i - z_j} (D_i - D_j) = \lambda \sum_{m,n=1}^{n+m \leq N} (m+n) p_m p_n \frac{\partial}{\partial p_{m+n}} + \lambda \sum_{n=1}^{N} n(N-n) p_n \frac{\partial}{\partial p_n}. \tag{19}$$

Note that neither m nor n is allowed to be zero in the first sum on the right. The excluded terms—those at the ends, without 2's, in the z series—appear as the nN part of the second sum. The $-n^2$ part arises from the restriction that i cannot equal j.

The $n+m \leq N$ constraint in the first sum on the right in (19) is natural because only p_n's with $n \leq N$ are algebraically independent, so the wavefunction, when expressed in terms of the p_n's, should not contain p_n's with $n > N$. Correspondingly, any p_{n+m} with $n + m > N$ generated by an application of the operator in (17) should, in principle, be re-expressed in terms of p_n's with $n \leq N$ by means of the Newton-Girard relations. It is, however, not unreasonable to ignore these issues in the collective field formalism. This is because we are ultimately interested in taking a thermodynamic limit in which we simultaneously rescale the mass of particles and the circumference of the circle so as to let $N \to \infty$ while keeping the physical density and non-relativistic dispersion fixed.

If we ignore the $n + m \leq N$ constraint and allow the sums to extend to infinity, we have

$$2H' = \sum_{n,m=1}^{\infty} \left(nm p_{n+m} \frac{\partial}{\partial p_n} \frac{\partial}{\partial p_m} + \lambda (m+n) p_m p_n \frac{\partial}{\partial p_{m+n}} \right) + \sum_{n=1}^{\infty} \left((1-\lambda) n^2 + \lambda n N \right) p_n \frac{\partial}{\partial p_n}. \tag{20}$$

Now H' should be Hermitian, and the right-hand-side of (20) is manifestly so if

$$p_n^\dagger = \frac{n}{\lambda} \frac{\partial}{\partial p_n}. \tag{21}$$

We know, from standard chiral bosonization [19], that this identification is correct for $\lambda = 1$. Accepting the identification for general λ, we can evaluate the inner product

$$\langle p_{\{\alpha\}} | p_{\{\beta\}} \rangle_{\text{Jack}} \equiv \langle p_1^{m_1} \cdots p_N^{m_N} | p_1^{n_1} \cdots p_N^{n_N} \rangle_{\text{Jack}}$$

$$
\begin{aligned}
&= \langle 1|(p_N^\dagger)^{m_N}\cdots(p_1^\dagger)^{m_1}p_1^{n_1}\cdots p_N^{n_N}\rangle_{\rm Jack} \\
&= \delta_{\{\alpha\}\{\beta\}}\lambda^{-l\{\alpha\}}\prod_{i=1}^{N} m_i!\, i^{m_i}. \qquad (22)
\end{aligned}
$$

Here we are using a slightly different parametrization of the partitions: $\{\alpha\}\equiv\{1^{m_1}2^{m_2}\ldots\}$, where the integer m_i is the number of rows in the Young diagram of $\{\alpha\}$ containing i boxes, and so the length of the partition is given by $l(\{\alpha\})\equiv m_1+m_2+\cdots$. The expression (22) for the inner product of the $p_{\{\alpha\}}$ defines what we will call the "λ-Jack" inner product. It can be expressed in Bargmann-Fock integral form as

$$
\langle F(p)|G(p)\rangle_{\rm Jack} = \int\prod_{n=1}^{\infty}\left(\lambda\frac{d^2p_n}{\pi n}\right)[F(p)]^*G(p)\exp\left\{-\lambda\sum_{n=1}^{\infty}\frac{1}{n}p_n^*p_n\right\}, \qquad (23)
$$

where $d^2p_n = d[\mathrm{Re}\,p_n]\,d[\mathrm{Im}\,p_n]$ and each integration is over the entire complex p_n plane.

Macdonald [16] uses this new inner product to define the Jack polynomials by again applying the Gramm-Schmidt procedure to the monomial symmetric functions $m_{\{\alpha\}}(z)$. Now the λ-Sutherland and the λ-Jack inner products are in general different. They only coincide when $\lambda=1$ (this is the miracle behind conventional bosonization) or when N is infinite. Remarkably, however, the Gramm-Schmidt procedure yields the same polynomials whichever product is used. This is because the Jack polynomials are mutually orthogonal with respect to *both* inner products, although their norms differ.

If, for $n>0$, we set $j_n=p_n$, $j_{-n}=j_n^\dagger=p_n^\dagger$ and $\nu=\lambda^{-1}$, we have the filling-fraction ν chiral algebra

$$
[j_n, j_m] = \nu m\,\delta_{n+m,0}. \qquad (24)
$$

We also set $j_0=N/2$, anticipating that the other $N/2$ will go in the left-going current. In position space

$$
j(\theta) = \frac{1}{2\pi}\sum_{n=-N}^{N} j_n e^{-in\theta}, \qquad (25)
$$

and the current algebra becomes

$$
[j(\theta), j(\theta')] = -\frac{i\nu}{2\pi}\delta'(\theta-\theta'), \qquad (26)
$$

which is the familiar right-going current commutator, at least at $\nu=1$.

In terms of the current components j_n we can write

$$
2H' = \lambda^2\sum_{n,m=1}^{\infty}(j_{n+m}j_{-n}j_{-m}+j_nj_mj_{-n-m}) + \lambda\sum_{n=1}^{\infty}((1-\lambda)n+\lambda N)\,j_nj_{-n}, \qquad (27)
$$

which is manifestly Hermitian with respect to the λ-Jack inner product, and normal-ordered. In position space the cubic terms in $2H'$ become

$$4\pi^2 \int_0^{2\pi} \frac{\lambda^2}{3} j(\theta)^3 d\theta, \tag{28}$$

where normal-ordering is to be understood. The quadratic terms can be written as an integral of a periodic Hilbert transform

$$[\varphi(\theta)]_{\mathrm{H}} \stackrel{\mathrm{def}}{=} \frac{1}{2\pi} P \int_0^{2\pi} \varphi(\theta') \cot\left(\frac{\theta-\theta'}{2}\right) d\theta' \tag{29}$$

for which $(e^{-in\theta})_{\mathrm{H}} = i\,\mathrm{sgn}\,(n) e^{-in\theta}$. We find that

$$\begin{aligned} 2H' &= 4\pi^2 \int_0^{2\pi} \left\{ \frac{\lambda^2}{3} j(\theta)^3 - iaj(\theta)\partial_\theta (j_+(\theta) - j_-(\theta)) \right\} d\theta, \\ &= 4\pi^2 \int_0^{2\pi} \left\{ \frac{\lambda^2}{3} j^3 - aj\partial_\theta j_{\mathrm{H}} \right\} d\theta. \end{aligned} \tag{30}$$

Here

$$a = \lambda(\lambda-1)/4\pi, \tag{31}$$

and j_+ is the part of j with j_n, $n>0$, and similarly j_- has j_n with $n<0$.

The resulting classical (where $\lambda(\lambda-1) \to \lambda^2$, because the "1" is really an $\hbar$) equation of motion is of Benjamin-Ono form

$$\partial_\tau j + j\partial_\theta j - \beta\, \partial^2_{\theta\theta} j_{\mathrm{H}} = 0, \tag{32}$$

where $\tau = 2\pi\lambda t$, and $\beta = 1/4\pi$. Seen from a frame moving at the speed of sound $c = \pi\lambda\rho_0$—so as to remove the convective effect of the constant background $\langle j \rangle = \rho_0/2$ —the Benjamin-Ono equation on the infinite line has a right-going solition solution

$$j(x,t) - \langle j \rangle = \frac{4U}{\beta^{-2}U^2[x-U\tau]^2+1}. \tag{33}$$

Here $2\pi\lambda U = (v_{\mathrm{soliton}} - c)$ must be positive, so the solitons always travel faster than the speed of sound [28]. The excess charge carried by the solition is

$$\int_{-\infty}^{\infty} (j(x,t) - \langle j \rangle)\, dx = 4\pi\beta = 1. \tag{34}$$

This solution is close to, but not identical with, the soliton solution for the continuum approximation to the classical Calogero model found by Polychronakos [13]. The difference is that Polychronakos' solitions can travel both to the left and right, and the width of his soliton is $\lambda c/[v^2_{\mathrm{soliton}} - c^2]$. The present soliton width is $\lambda/[2(v_{\mathrm{soliton}} - c)]$. The two widths coincide, however, when $v_{\mathrm{soliton}} - c$ is small compared to c, *i.e.* when the excitation momentum is small compared to the distance between the left and right Fermi surfaces.

IV. THE INNER PRODUCTS AND THE COLLECTIVE-FIELD MEASURE

The Jack polynomials form an orthogonal, but not orthonormal, basis for the symmetric functions with respect to both the Jack and Sutherland inner products.

We have [16]

$$\langle J_{\{\alpha\}} | J_{\{\alpha\}} \rangle_{\text{Sutherland}} = \prod_{1 \le i < j \le N} \frac{\Gamma(\xi_i - \xi_j + \lambda)\Gamma(\xi_i - \xi_j - \lambda + 1)}{\Gamma(\xi_i - \xi_j)\Gamma(\xi_i - \xi_j + 1)}, \tag{35}$$

where the ξ_i associated with the partition $\{\alpha\}$ are

$$\xi_i = \alpha_i + \lambda k_i^0 \tag{36}$$

are the pseudomomenta,in terms of which the Sutherland energy eigenvalue is

$$E_{\{\alpha\}} = \frac{1}{2} \sum_{i=1}^{N} \xi_i^2.$$

The Jack product, on the other hand, gives [16]

$$\langle J_{\{\alpha\}} | J_{\{\alpha\}} \rangle_{\text{Jack}} = \prod_{s \in \{\alpha\}} \frac{a(s) + \lambda l(s) + 1}{a(s) + \lambda l(s) + \lambda}. \tag{37}$$

Here s labels a box in the Young diagram of the partition $\{\alpha\}$, and $a(s)$ and $l(s)$ are respectively the *arm length* (the number of boxes to the right of s) and *leg length* (the number of boxes below s) of s.

The relation between the two norms is [16]

$$\langle J_{\{\alpha\}} | J_{\{\alpha\}} \rangle_{\text{Sutherland}} = C_N \langle J_{\{\alpha\}} | J_{\{\alpha\}} \rangle_{\text{Jack}} \prod_{s \in \{\alpha\}} \frac{\lambda N + a'(s) - \lambda l'(s)}{\lambda N + a'(s) + 1 - \lambda(l'(s) + 1)}, \tag{38}$$

where $a'(s)$ and $l'(s)$ are respectively the *arm co-length* (the number of boxes to the left of s) and *leg co-length* (the number of boxes above s) of s, and

$$C_N = \langle 1 | 1 \rangle_{\text{Sutherland}} = \frac{1}{N!} \int_0^{2\pi} \cdots \int_0^{2\pi} \prod_{i=1}^{N} \frac{d\theta_i}{2\pi} |\Delta|^{2\lambda} = \frac{1}{N!} \frac{\Gamma(1 + \lambda N)}{[\Gamma(1 + \lambda)]^N}. \tag{39}$$

Inspection of (38) shows that scaled product $C_N^{-1} \langle \ldots | \ldots \rangle_{\text{Sutherland}}$ will coincide with the Jack product when $\lambda = 1$ or when $N \to \infty$ with all $a'(s)$ and $l'(s)$ remaining finite.

The source of the difficulty in decoupling the left and right Fermi-surface physics is that scaled Sutherland product need *not* coincide with the Jack product when $N \to \infty$ and at the

same time the number of rows or columns in the Young diagram remains $O(N)$. The former is exactly the situation when we when we seek to describe excitations near the left-hand Fermi surface. We do this exploiting the identity

$$(m_{\{1^N\}})^p J_{\{\alpha_1,\alpha_2,\dots,\alpha_N\}}(z) = J_{\{\alpha_1+p,\alpha_2+p,\dots,\alpha_N+p\}}(z), \tag{40}$$

where $m_{\{1^N\}}(z) = z_1 z_2 \cdots z_N$, to add p columns of N boxes on the left of the Young diagram representing the Sutherland eigenstate. This operation [17] corresponds to a Galilean boost in which each of the N particles is given an additional p quanta of momentum. If p is made large enough, all the particle momenta can be made positive. The Sutherland inner product (but not the Jack product) is invariant under such boosts. We can create negative-momentum excitations near the left-hand Fermi surface by removing boxes near the bottom of the, now N-row deep, Young diagram. This means that the left-most pseudo-momenta were not boosted quite as far as the others. When the boost is undone by removing the added columns, we are left with a Young diagram with some negative-length rows. The corresponding H' eigenfunction is now a rational function rather than a polynomial, but it can be written as a conventional Jack polynomial multiplied by a negative power of $m_{\{1^N\}}$. The Jack and Sutherland products will not coincide for such ambichiral states.

To understand the consequences of this difference between the Sutherland and Jack products in the collective field language, we begin by exploring how it is that these rather differently defined products become equal in the large-N chiral case.

If $|z_i| = 1$, and $|\mu| < 1$ is a convergence factor inserted to make the logarithmic series converge, we have

$$\begin{aligned}
\exp\left\{-\lambda \sum_{n=1}^{\infty} \frac{1}{n}\mu^n p_n^* p_n\right\} &= \prod_{i,j}(1-\mu z_i^* z_j)^\lambda \\
&= \prod_{i<j}(1-\mu z_i^* z_j)^\lambda (1-\mu)^{N\lambda}(1-\mu z_j^* z_i)^\lambda \\
&= \prod_{i<j}(z_i-\mu z_j)^\lambda (z_i^*-\mu z_j^*)^\lambda (1-\mu)^{N\lambda} \\
&\to |\Delta(z)|^{2\lambda}(1-\mu)^{N\lambda}, \quad \text{as } \mu \to 1_-.
\end{aligned} \tag{41}$$

We see that the explicit weights in the Sutherland and Jack products are in some sense proportional, but the constant of proportionality diverges to zero as $\mu \to 1$.

We do not need a convergence factor in

$$\sum_{n=-\infty}^{\infty} \frac{1}{|n|} e^{in\theta} = -2 \ln |2 \sin(\theta/2)|, \tag{42}$$

and so with $\rho(\theta) = \frac{1}{2\pi} \sum p_n e^{-in\theta}$ we have

$$\exp\left\{-\lambda \sum_{n=1}^{\infty} \frac{1}{n} p_n p_{-n}\right\} = \exp\left\{\lambda \int_0^{2\pi} \int_0^{2\pi} \rho(\theta)\rho(\theta') \ln \left|2 \sin\left(\frac{\theta-\theta'}{2}\right)\right| d\theta d\theta'\right\}. \tag{43}$$

An additive constant p_0 in ρ does not contribute to the right-hand side because the kernel integrates to zero. The singularity in the integrand is integrable. What does this mean for the divergent "$i = j$" factors in the exact product? Should the integral contain a counterterm to remove them? The appropriate replacement is [20]

$$\prod_{i,<j} |z_i - z_j|^{2\lambda} \simeq C \exp\left\{\lambda \int_0^{2\pi} \rho(\theta) \ln \rho(\theta)\, d\theta + \lambda \int_0^{2\pi} \int_0^{2\pi} \rho(\theta)\rho(\theta') \ln \left|2 \sin\left(\frac{\theta-\theta'}{2}\right)\right| d\theta d\theta'\right\}. \tag{44}$$

The first term subtracts a ln(interparticle spacing) self-energy for each particle, and is consistent with the observation that when the z_i are equally spaced round the unit circle we

$$\prod_{i<j} |z_i - z_j|^{2\lambda} \to (N^{N/2})^{2\lambda} = \exp\{\lambda N \ln N\}. \tag{45}$$

In a "coulomb gas" interpretation the first term in the exponent in (44) computes the microscopic internal energy of the uniform gas, and the second accounts for the electrostatic energy due to macroscopic deviations from uniformity.

In addition to expressing the $|\Delta|^{2\lambda}$ weight in terms of the particle density, we need to compute the Jacobian of the transformation from the z_i to the p_n. This change of variable is conceptually subtle. The map $(z_1, \ldots, z_N) \to (p_1, \ldots, p_N)$ is not invertible: each of the z_i has unit modulus, whilst in the Bargmann-Fock integral the p_n are general complex numbers. An arbitrary set of p_n will not arise from from z_i with $|z_i| = 1$. However, as the z_i move on their unit circles, each p_n moves as the endpoint of an N-step random walk in the complex plane with $\langle |p_n|^2 \rangle = N$. By the central limit theorem, therefore, each p_n has large-N probability density

$$P(p_n) = (N\pi)^{-1} e^{-|p_n|^2/N}. \tag{46}$$

It is natural to conjecture that as $N \to \infty$ the map $z_i \to z_i^n$ so scrambles the directions of the individual z_i^n steps that their sums $p_n = \sum_i z_i^n$ become *independent* random variables

with joint probability density

$$P(p_1, p_2, \ldots) \propto \exp\left\{-\frac{1}{N}\sum_{n=1}^{\infty} p_n p_{-n}\right\} = \exp\left\{-\frac{1}{2\rho_0}\int_0^{2\pi} (\rho'(\theta))^2 \, d\theta\right\}. \tag{47}$$

Here $\rho' = \rho - \rho_0$ and $\rho_0 = N/2\pi$. As N becomes large this distribution becomes uniform on the scale of the early ($n \ll N$) exponentials in the λ-Jack Bargmann-Fock interal and so the low-momentum integration measures in the Sutherland and Jack products are also proportional— despite one integration domain having twice the dimension as the other. (In other words the large-N image of the real N-torus is dense in C^N.)

The integration measure will not appear uniform if applied to wavefunctions containing p_n's with $n = O(N)$. In this case we need a more accurate formula. Jevicki shows [21] that our conjectured probability density (47) is but the first term in a systematic expansion in powers of $1/N$:

$$P(p_1, p_2, \ldots) \propto \exp\left\{\int_0^{2\pi}\left[-\frac{1}{2\rho_0}\rho'^2 + \frac{1}{6\rho_0^2}\rho'^3 - \frac{1}{12\rho_0^3}\rho'^4 + \cdots\right] d\theta\right\}. \tag{48}$$

Now we observe that

$$\int_0^{2\pi}\left[-\frac{1}{2\rho_0}\rho'^2 + \frac{1}{6\rho_0^2}\rho'^3 - \frac{1}{12\rho_0^3}\rho'^4 + \cdots\right] d\theta = \int_0^{2\pi} \left[\rho_0 \ln \rho_0 - (\rho_0 + \rho') \ln(\rho_0 + \rho')\right] \, d\theta, \tag{49}$$

and so surmise that

$$P(p_1, p_2, \ldots) \propto \exp\left\{-\int_0^{2\pi} \rho \ln \rho \, d\theta\right\}. \tag{50}$$

To verify this conjecture, we can proceed as follows: we want to find the measure $P[p_n]$ such that

$$\int_0^{2\pi} \cdots \int_0^{2\pi} \prod_{i=1}^{N} \frac{d\theta_i}{2\pi} F\left(\sum_i e^{in\theta_i}\right) = \int \prod_{n=1}^{\infty} dp_n P[p_n] F(p_n). \tag{51}$$

Let $\lambda(\theta) = \sum_n \lambda_n e^{in\theta}$, and, as usual, $\rho(\theta) = \frac{1}{2\pi}\sum_n p_n e^{-in\theta}$. Thus

$$\begin{aligned} P[p_n] &= \int_0^{2\pi} \cdots \int_0^{2\pi} \prod_{i=1}^{N} \frac{d\theta_i}{2\pi} \prod_{n=1}^{\infty} \delta\left(-p_n + \sum_i e^{in\theta_i}\right) \\ &= \int_0^{2\pi} \cdots \int_0^{2\pi} \prod_{i=1}^{N} \frac{d\theta_i}{2\pi} \int \prod_{n=1}^{\infty} d\lambda_n \exp\left\{\lambda_n\left(-p_n + \sum_i e^{in\theta_i}\right)\right\} \\ &= \int d[\lambda(\theta)] \exp\left\{-\int d\theta \lambda(\theta)\rho(\theta)\right\}\left[\int \frac{d\theta}{2\pi} \exp \lambda(\theta)\right]^N . \end{aligned} \tag{52}$$

We now introduce a chemical potential μ. We multiply the last line by $(2\pi)^N \exp(N\mu)/N!$ and sum over N. This gives

$$P[\rho] = \int d[\lambda(\theta)] \exp\left\{ \int d\theta[-\lambda(\theta)\rho(\theta) + \exp(\lambda(\theta) + \mu)] \right\} \tag{53}$$

The value of μ will be chosen so as to enforce $\int \rho\, d\theta = N$. In the thermodynamic limit there should be no difference between the canonical and grand canonical ensembles.

Next, the $\lambda(\theta)$ functional integral is approximated by stationary phase. Calling the exponent $S[\lambda, \rho]$, we have

$$\delta S = \int_0^{2\pi} \delta\lambda(\theta) \left(-\rho(\theta) + \exp(\lambda(\theta) + \mu)\right) \, d\theta, \tag{54}$$

Thus

$$\lambda_{\text{stationary}}(\theta) = \ln \rho(\theta) - \mu, \tag{55}$$

and

$$P[\rho] \sim \exp\left\{ S[\lambda_{\text{stationary}}, \rho] \right\} = \exp\left\{ \int_0^{2\pi} (-\rho \ln \rho + \rho + \mu\rho) d\theta \right\}. \tag{56}$$

Corrections to the leading-order stationary-phase result are also in powers of $1/N$, but they have a different character from the $1/N$ corrections inherent in $\rho \ln \rho$. The $\lambda(\theta)$ functional integral is ultra-local, and so the coefficients will involve $\delta(0)$'s [21]. These divergent terms must compensate for divergences arising in the resulting continuum $\rho(\theta)$ field theory. The underlying Schrödinger problem, after all, has no divergences.

The $-\rho \ln \rho$ in the exponent of the measure makes physical sense. It is the configurational entropy of the non-uniform gas. The number of ways of distributing the N distinguishable particles (they are labelled by the "i" on θ_i) into k bins of length $2\pi/k$, with n_1 in bin 1, n_2 in bin 2, *etc.*, is

$$\begin{aligned} \frac{N!}{n_1! n_2! \cdots n_k!} &\approx \exp\left\{ N \ln N - N - \sum_{\alpha=1}^{k} (n_\alpha \ln n_\alpha - n_\alpha) \right\}, \\ &\approx \exp\left\{ \int_0^{2\pi} (-\rho \ln \rho + \rho + \text{const.}) d\theta \right\}. \end{aligned} \tag{57}$$

The steepest descent approximation to the integral over $\lambda(\theta)$ is now seen to be the steepest descent approximation that gives Stirling's approximation:

$$\begin{aligned} \frac{1}{n!} = \frac{1}{\Gamma(n+1)} &= \frac{1}{2\pi i} \int_C t^{-(n+1)} e^t \, dt, \\ &= \frac{1}{2\pi i} \int_{C'} \exp\left\{ -n\lambda + e^\lambda \right\} d\lambda, \\ &\approx \exp\left\{ -n \ln n + n \right\}. \end{aligned} \tag{58}$$

In the second line we have set $t = \exp\lambda$ and in the last line approximated the integral by the maximum value of its integrand, which occurs at $\lambda = \ln n$.

In conclusion, we have that the Sutherland product integral

$$\int_0^{2\pi} \cdots \int_0^{2\pi} \prod_{i=1}^{N} \frac{d\theta_i}{2\pi} |\Delta|^{2\lambda} \ldots \tag{59}$$

becomes, in the collective field formalism, proportional to a functional integral over $\rho(\theta)$ with weight [22, 23, 24]

$$J[\rho] = \exp\left\{ (\lambda - 1) \int_0^{2\pi} \rho(\theta) \ln \rho(\theta)\, d\theta + \lambda \int_0^{2\pi} \int_0^{2\pi} \rho(\theta)\rho(\theta') \ln \left| 2\sin\left(\frac{\theta - \theta'}{2}\right)\right| \, d\theta d\theta' \right\}. \tag{60}$$

The first term in the exponent is absent in the collective-field form of the Jack inner product.

V. INCORPORATING THE LEFT-GOING MODES

In the purely right-going case the wavefunction depended only on the p_n for n positive, and p_{-n} was interpreted as the Bargmann-Fock adjoint of the operation of multiplication by p_n. To decribe both left- and right-going excitations simultaneously we have to allow wavefunctions contining both p_n and p_{-n}. These complex variables should be conjugates of each other, and so the independent variables are their real and imaginary parts r_n, s_n with $n > 0$. Thus $p_n = r_n + is_n$ and $p_{-n} = r_n - is_n$, and

$$\frac{\partial}{\partial p_n} = \frac{1}{2}\left(\frac{\partial}{\partial r_n} - i\frac{\partial}{\partial s_n}\right), \quad \frac{\partial}{\partial p_{-n}} = \frac{1}{2}\left(\frac{\partial}{\partial r_n} + i\frac{\partial}{\partial s_n}\right), \quad n > 0. \tag{61}$$

Let us begin by taking the inner product to have the Jack-product weight

$$J = \exp\left\{ -\lambda \sum_{n=1}^{\infty} \frac{1}{n} p_n p_n^* \right\} = \exp\left\{ -\lambda \sum_{n=1}^{\infty} \frac{1}{n}(r_n^2 + s_n^2) \right\}. \tag{62}$$

Then, with respect to this new, non-chiral Jack product—let's call it Jack′—we have

$$\left(\frac{\partial}{\partial r_n}\right)^\dagger = -\frac{1}{J}\frac{\partial}{\partial r_n} J = -\frac{\partial}{\partial r_n} + \frac{2\lambda}{n} r_n, \tag{63}$$

and similarly for s_n. Proceeding in this manner we find that

$$\left(\frac{\partial}{\partial p_n}\right)^\dagger = -\frac{\partial}{\partial p_{-n}} + \frac{\lambda}{n}\mathrm{sgn}\,(n) p_n, \tag{64}$$

where n can have either sign. Also $p_n^\dagger = p_{-n}$. We note that $(\ldots)^{\dagger\dagger} = (\ldots)$.

Now define

$$\mathrm{v}_n = 2\pi \left(-n \frac{\partial}{\partial p_{-n}} + \frac{\lambda}{2} \mathrm{sgn}\,(n) p_n \right), \tag{65}$$

so that $\mathrm{v}_n^\dagger = \mathrm{v}_{-n}$ as the Jack′-product adjoint. With

$$\mathrm{v}(\theta) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \mathrm{v}_n e^{-in\theta}, \quad \rho(\theta) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} p_n e^{-in\theta}, \tag{66}$$

both $\mathrm{v}(\theta)$ and $\rho(\theta)$ are Jack′ hermitian, and

$$[\rho(\theta), \mathrm{v}(\theta')] = -i\hbar \partial_\theta \delta(\theta - \theta'). \tag{67}$$

We now define chiral currents $j_{\mathrm{R,L}} = \frac{1}{2}(\rho \pm \mathrm{v}/\pi\lambda)$ with

$$\begin{aligned} j_{\mathrm{R}.n} &= -\frac{n}{\lambda} \frac{\partial}{\partial p_{-n}} + \Theta(n) p_n, \\ j_{\mathrm{L}.n} &= +\frac{n}{\lambda} \frac{\partial}{\partial p_{-n}} + \Theta(-n) p_n. \end{aligned}$$

These obey

$$\begin{aligned} [j_{\mathrm{R},n}, j_{\mathrm{R},m}] &= \quad m\nu \delta_{m+n,0}, \\ [j_{\mathrm{L},n}, j_{\mathrm{L},m}] &= -m\nu \delta_{m+n,0}, \\ [j_{\mathrm{L},n}, j_{\mathrm{R},m}] &= \quad 0, \end{aligned}$$

and so the right and left current algebras are cleanly separated. We should take $\Theta(0) = \frac{1}{2}$ so as to agree with our previous allocation of the half of $p_0 = N$ to each of the chiral currents. Unlike the chiral case, the left- and right-going currents have p_n derivatives containing *both* signs of n.

To write the Hamiltonian in terms of the extended set of p_n we need

$$(z_i + z_j) \frac{z_i^{-n} - z_j^{-n}}{z_i - z_j} = -(z_i^{-n} + 2 z_i^{-n+1} z_j^{-1} + \cdots + 2 z_i^{-1} z_j^{-n+1} + z_j^{-n}) \tag{68}$$

in addition to our previous

$$(z_i + z_j) \frac{z_i^n - z_j^n}{z_i - z_j} = z_i^n + 2 z_i^{n-1} z_j + \cdots + 2 z_i z_j^{n-1} + z_j^n. \tag{69}$$

The hamiltonian becomes [25]

$$\begin{aligned} 2H' = & \sum_{n,m=-\infty}^{\infty} nm p_{n+m} \frac{\partial}{\partial p_n} \frac{\partial}{\partial p_m} + \lambda \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (n+m) \left(p_n p_m \frac{\partial}{\partial p_{n+m}} + p_{-n} p_{-m} \frac{\partial}{\partial p_{-n-m}} \right) + \\ & + \sum_{n=-\infty}^{\infty} \left((1-\lambda) n^2 + \lambda |n| N \right) p_n \frac{\partial}{\partial p_n}. \end{aligned} \tag{70}$$

Now the (normal-ordered) expression

$$4\pi^2 \int_0^{2\pi} \left(\frac{\lambda^2}{3} j_{\mathrm{R}}^3 + \frac{\lambda^2}{3} j_{\mathrm{L}}^3 \right) d\,\theta$$

is equal to

$$\sum_{n,m=-\infty}^{\infty} nm p_{n+m} \frac{\partial}{\partial p_n} \frac{\partial}{\partial p_m} \;+\; \lambda \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (n+m) \left(p_n p_m \frac{\partial}{\partial p_{n+m}} + p_{-n} p_{-m} \frac{\partial}{\partial p_{-n-m}} \right) + \sum_{n=-\infty}^{\infty} \lambda |n| N p_n \frac{\partial}{\partial p_n},$$

and the total momentum is

$$\begin{aligned} \hat{P}_{\mathrm{tot}} \equiv \sum_i D_i = \sum_{n=-\infty}^{\infty} n p_n \frac{\partial}{\partial p_n} \;&=\; \frac{\lambda}{2} \sum_{n=-\infty}^{\infty} \left(j_{\mathrm{R},n} j_{\mathrm{R},-n} - j_{\mathrm{L},-n} j_{\mathrm{L},n} \right) \\ &= \lambda \pi \int_0^{2\pi} (j_{\mathrm{R}}^2 - j_{\mathrm{L}}^2)\, d\theta = \int_0^{2\pi} \rho \mathrm{v}\, d\theta. \end{aligned}$$

In the momentum, the unwanted terms with two $\partial/\partial p_n$'s cancelled between the left- and right-going current contributions..

Life seems more complicated if we wish to assert that the remaining term in $2H'$

$$\sum_{n=-\infty}^{\infty} n^2 p_n \frac{\partial}{\partial p_n} \stackrel{?}{=} \lambda \sum_{n=1}^{\infty} n \left(j_{\mathrm{R},n} j_{\mathrm{R},-n} + j_{\mathrm{L},-n} j_{\mathrm{L},n} \right).$$

Here, although the $p_n \partial/\partial p_n$ terms are generated correctly, the undesired two-derivative terms appearing in the right-hand-side do <u>not</u> cancel. Even worse, we find that while $\hat{P}_{\mathrm{tot}}$ is Hermitian thanks to cancellations between terms with $\pm n$, the expression $\sum n^2 p_n \partial/\partial p_n$ is *not* Hermitian with respect to the Jack$'$ inner product. This means that, while in the chiral case H' was Hermitian with respect to both the Sutherland and Jack products, in the non-chiral case it is Hermitian only with respect to the Sutherland product.

A further indication that the Sutherland product is essential comes from realizing that $\mathrm{v}(\theta)$ is *not* the physical velocity field. From number conservation

$$\dot{\rho} + \partial_\theta \rho v = 0$$

and the Heisenberg equation of motion $\dot{\rho} = i[H', \rho]$, we should have $[H', \rho] = i\partial_\theta(\rho v)$, or, in Fourier space,

$$n(\rho v)_n = [H, \rho_n], \quad \text{where} \quad n(\rho v)_n \equiv \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} n p_{n-m} v_m.$$

Now

$$\sum_{n,m=-\infty}^{\infty} p_n p_m (n+m) \mathrm{sgn}\,(m) \frac{\partial}{\partial p_{n+m}}$$

$$= \sum_{n,m=1}^{\infty} (n+m) \left(p_n p_m \frac{\partial}{\partial p_{n+m}} \ + \ p_{-n} p_{-m} \frac{\partial}{\partial p_{-n-m}} \right) + \sum_{m=-\infty}^{\infty} N|m| p_m \frac{\partial}{\partial p_m}, \quad (71)$$

since the terms with n and m of opposite sign cancel, and the last term comes about because m cannot be zero ($\mathrm{sgn}\,(0) = 0$) but n *can* be zero, and $p_0 = N$. Using (71) we find that

$$\begin{aligned} [H', \rho_n] &= \sum_{m=-\infty}^{\infty} n \left(m p_{n+m} \frac{\partial}{\partial p_m} + \frac{\lambda}{2} p_{n-m} \mathrm{sgn}\,(m) p_m \right) + \tfrac{1}{2}(1-\lambda) n^2 p_n \\ &= \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} n p_{n-m} \left[2\pi \left(-m \frac{\partial}{\partial p_{-m}} + \frac{\lambda}{2} \mathrm{sgn}\,(m) p_m \right) \right] + \tfrac{1}{2}(1-\lambda) n^2 p_n \\ &= \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} n p_{n-m} \mathrm{v}_m + \tfrac{1}{2}(1-\lambda) n^2 p_n \end{aligned}$$

The first term contains our v_m's, and the last is the Fourier transform of

$$i\partial_\theta \left\{ \rho \frac{i}{2}(1-\lambda) \partial_\theta \ln \rho \right\}.$$

We conclude that

$$v_{\mathrm{physical}} = \mathrm{v} + \frac{i}{2}(1-\lambda)\partial_\theta \ln \rho.$$

We note, however, the comforting fact that

$$\hat{P}_{\mathrm{tot}} \equiv \sum_i D_i = \int_0^{2\pi} \rho \mathrm{v}\, d\theta = \int_0^{2\pi} \rho v_{\mathrm{physical}}\, d\theta,$$

because the addition to the momentum density is a total derivative.

The distinction between v and v_{physical} is accounted for by the different weights in the Jack and Sutherland products. To see this, we work in position space. As usual we have

$$\rho(\theta) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} p_n e^{-in\theta}.$$

We define

$$\Pi(\theta) \equiv \frac{\delta}{\delta\rho(\theta)} = \sum_{n=-\infty}^{\infty} e^{-in\theta} \frac{\partial}{\partial p_{-n}} \quad (72)$$

so that $\delta\rho(\theta)/\delta\rho(\theta') = \delta(\theta - \theta')$. The operator

$$-i\partial_\theta \Pi = \sum_{n=-\infty}^{\infty} \left(-n \frac{\partial}{\partial p_{-n}} \right) e^{-in\theta} \quad (73)$$

is hermitian with respect to a product defined by an integration over ρ with weight unity. If we let

$$K[\rho] = \exp\left\{\lambda \int_0^{2\pi}\int_0^{2\pi} \rho(\theta)\rho(\theta')\ln\left|2\sin\left(\frac{\theta-\theta'}{2}\right)\right| d\theta d\theta'\right\} \tag{74}$$

be the weight appearing in the Jack product, then our

$$\mathrm{v}(\theta) = K^{-1/2}(-i\partial_\theta \Pi)K^{1/2} = -i\partial_\theta\Pi - i\frac{\lambda}{2}\rho_{\mathrm{H}} \tag{75}$$

is Jack′ Hermitian, and

$$v_{\text{physical}} = J^{-1/2}(-i\partial_\theta\Pi)J^{1/2} = -i\partial_\theta\Pi - i\frac{\lambda}{2}\rho_{\mathrm{H}} - i\frac{(\lambda-1)}{2}\partial_\theta \ln\rho \tag{76}$$

is Hermitian with respect to the Sutherland product, which contains the weight $J[\rho]$.

Now let us return to problem of expressing the remaining sum in the Hamiltonian in terms of physical variables. We note that

$$\sum_{n=-\infty}^{\infty} \operatorname{sgn}(n) n^2 p_n \frac{\partial}{\partial p_n} \;=\; \lambda \sum_{n=1}^{\infty} n\left(j_{\mathrm{R},n} j_{\mathrm{R},-n} - j_{\mathrm{L},-n} j_{\mathrm{L},n}\right),$$

is Jack′ hermitian, but this is not quite the expression that appears in the Hamiltonian. We need to remove the sgn(n) by changing the sign of the negative n terms in this sum. The following manouvre, a paraphrase of the unusual Hilbert transform in [14], achieves this. We start with

$$\begin{aligned} 4\pi^2\int_0^{2\pi}\left\{\frac{\lambda^2}{6}j_{\mathrm{R}}^3 + \frac{\lambda^2}{6}j_{\mathrm{L}}^3\right\}d\theta \;&=\; \int_0^{2\pi}\left\{\frac{\lambda^2\pi^2}{6}\rho^3 + \frac{1}{2}\rho\mathrm{v}^2\right\}d\theta \\ &=\; \int_0^{2\pi}\left\{\frac{\lambda^2\pi^2}{6}\rho^3 + \frac{1}{2}\rho v_{\text{phys}}^2 + \frac{i}{2}(\lambda-1)v_{\text{phys}}\partial_\theta\rho - \frac{1}{8}\frac{(\partial_\theta\rho)^2}{\rho}\right\}d\theta \end{aligned}$$

and observe that adding

$$\begin{aligned} &\int_0^{2\pi}\left\{\frac{i}{4}(\lambda-1)\left[j_{\mathrm{R}}\partial_\theta(\mathrm{v}+2\pi i\lambda\rho_{\mathrm{H}} - \pi\lambda\rho) + j_{\mathrm{L}}\partial_\theta(\mathrm{v}+2\pi i\lambda\rho_{\mathrm{H}}+\pi\lambda\rho)\right]\right\}d\theta \\ = &\int_0^{2\pi}\left\{-\frac{i}{2}(\lambda-1)v_{\text{phys}}\partial_\theta\rho + \frac{1}{4}(\lambda-1)^2\frac{(\partial_\theta\rho)^2}{\rho} - \frac{\pi}{2}\lambda(\lambda-1)\rho\partial_\theta\rho_{\mathrm{H}}\right\}d\theta \end{aligned}$$

gives the known position-space Hamiltonian [13]

$$H' = \int_0^{2\pi}\left\{\frac{1}{2}\rho v_{\text{physical}}^2 + \frac{\lambda^2\pi^2}{6}\rho^3 + \frac{1}{8}(\lambda-1)^2\frac{(\partial_\theta\rho)^2}{\rho} - \frac{\pi}{2}\lambda(\lambda-1)\rho\partial_\theta\rho_{\mathrm{H}}\right\}d\theta. \tag{77}$$

The shift $\mathrm{v} \to \mathrm{v} + 2\pi i\lambda\rho_{\mathrm{H}} \mp \pi\lambda\rho$ has changed the signs before the $\Theta(\pm n)$'s in the definitions of $j_{\mathrm{L,R}}$, and thus effected the desired change of sign of the negative n terms in the $n^2 p_n \partial/p_n$ sum.

Setting $j_{\mathrm{L}}(\theta) = 0$ in this Hamiltonian reduces it to an expression

$$H_{\text{constrained}} = 4\pi^2 \int_0^{2\pi} \left\{ \frac{\lambda^2}{6} j_{\mathrm{R}}^3 - \frac{\lambda(\lambda-1)}{8\pi} j_{\mathrm{R}} \partial (j_{\mathrm{R}})_{\mathrm{H}} \right\} d\theta. \tag{78}$$

that looks very like the chiral Hamiltonian appearing in (30). Further, by examining the $n > 0$ Fourier components in (68), we see that imposing $j_{\mathrm{L}}(\theta) = 0$ as a constraint on the wavefunctions is equivalent to demanding that

$$\frac{\partial}{\partial p_n} \to 0, \quad n < 0,$$

and so requires the wavefunction not depend on p_n with negative n. Equating the $n < 0$ Fourier components to zero requires that, as operators, we have

$$p_{-n} \to \frac{n}{\lambda} \frac{\partial}{\partial p_n}, \quad n > 0.$$

Consequently, p_{-n} ceases being independent and returns to being the Bargmann-Fock Jack-product adjoint of multiplication by p_n. We have precisely recovered the chiral theory from section III. The $j_{\mathrm{L}} = 0$ constraint, natural as it seems, is *not* however consistent with the full equations of motion: an initially-zero j_{L} does not remain zero.

The true, consistent, right-going chiral constraint was found by Bettelheim, Abanov and Wiegmann [15] to be

$$v_{\text{physical}} = \pi\lambda\rho - \frac{1}{2}(\lambda - 1)\partial_\theta (\ln\rho)_{\mathrm{H}}. \tag{79}$$

With this condition the separate continuity equation

$$\dot{\rho} + \partial_\theta\, \rho v_{\text{physical}} = 0, \tag{80}$$

and the Euler equation

$$\dot{v} + v_{\text{physical}} \partial_\theta v_{\text{physical}} = -\partial_\theta w(\rho), \tag{81}$$

where

$$w(\rho) = \frac{\lambda^2\pi^2\rho^2}{2} - \frac{(\lambda-1)^2}{8}\left(2\partial_{\theta\theta}^2 \ln\rho + (\partial_\theta \ln\rho)^2\right) - \pi\lambda(\lambda-1)\partial_\theta\rho_{\mathrm{H}}, \tag{82}$$

become identical —but only after some considerable algebra and use of Tricomi's version

$$(\phi_1(\phi_2)_{\mathrm{H}})_{\mathrm{H}} + ((\phi_1)_{\mathrm{H}}\phi_2)_{\mathrm{H}} = (\phi_1)_{\mathrm{H}}(\phi_2)_{\mathrm{H}} - \phi_1\phi_2. \tag{83}$$

of the Poincaré-Bertrand identity [26, 27]. The resulting single equation for the right-going wave is [15]

$$\dot{\rho} + \partial_\theta \left\{ \pi\lambda\rho^2 - \frac{1}{2}(\lambda-1)\rho\,\partial_\theta(\ln\rho)_{\mathrm{H}} \right\} = 0. \tag{84}$$

This equation can be made to coincide with our earlier Benjamin-Ono equation by linearizing $\rho\,\partial_\theta(\ln\rho)_{\mathrm{H}} \approx \partial_\theta\rho_{\mathrm{H}}$.

In terms of the current j_{L}, the rather mysterious chiral condition becomes

$$j_{\mathrm{L}} = i\frac{(\lambda-1)}{2\pi\lambda}\partial_\theta(\ln\rho)_- \tag{85}$$

Recall that the subscript "$-$" means a projection onto the $n<0$ Fourier modes. Therefore, from the $n>0$ Fourier components, we again read off that

$$\frac{\partial}{\partial p_n} \to 0, \quad n<0, \tag{86}$$

and the wavefunction remains only a function of the p_n for $n>0$. The $n<0$ components, however, now give

$$p_{-n} \to \frac{n}{\lambda}\frac{\partial}{\partial p_n} + i\frac{(\lambda-1)}{2\pi\lambda}\left[\partial_\theta(\ln\rho)\right]_{-n}, \quad n>0. \tag{87}$$

This equation asserts that that $p_{-n} = p_n^\dagger$ with the adjoint taken with respect to the *Sutherland* product. The true chiral condition is therefore a very natural, and indeed inevitable, consequence of the necessity of using only the Sutherland inner product when dealing with both the full ambichiral collective field.

VI. CONCLUSIONS

We have traced the difficulty in separating the left- and right-going degrees of freedom in the continuum hydrodynamics of the Sutherland model to the existence of two distinct inner products with respect to which the polynomial eigenfunctions are orthogonal. Each chiral half of the model is most naturally expressed in terms of operators that are hermitian with respect to the first of these products, but the full model is only hermitian with respect to the second.

We have still not managed to decouple the oppositely moving edge modes into non interacting waves, and it is an interesting question whether this is possible.

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